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Instantons and hypercontact structures

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Abstract

Any (anti)-instanton yields a set of three contact forms on S^7. The basic (anti)-instanton yields a hypercontact structure. Also, we indicate sufficient conditions for the components of an $SU(2)$ connection to be contact forms. Finally, we prove, under a mild hypothesis, that the three contact forms of any hypercontact structure define the same contact structure.

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1. Introduction

This note is a commentary on the recent observation by Geiges and Thomas [4] that the basic anti-instanton yields a hypercontact structure on S^7. Here we will discuss only instantons, since by reversing the orientation of the bundle, instantons become anti-instantons. Recall that an instanton is a self-dual connection on an $SU(2)$-bundle over S^4 with Pontryagin number $k = +1$, which we can assume to be the "tautological" bundle $\pi : S^7 \to S^4$. See [1,3,7].

Our first remark is that Geiges–Thomas' observation is transparent from the explicit formula of a "natural" connection on the "tautological" bundle whose potential over $S^4 - \{p\} \approx \mathbb{R}^4$ is

$$A(x) = \mathrm{Im}\left(\frac{x\,\mathrm{d}\bar{x}}{1+|x|^2}\right),$$

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where $p = (0, 0, 0, 0, 1)$ and $x \in \mathbb{R}^4$ is considered as a quaternionic variable. This is the basic instanton.

Our second remark is that Geiges–Thomas arguments to prove their observation actually allows to prove a more general result (Theorem 6).

Our final remark (Theorem 10) is that, under a mild hypothesis, the three contact forms of a hypercontact structure define the same contact structure. We prove that the barycentric path between any two of these contact forms is still a contact form and then apply the Gray–Martinet stability theorem [5].

We now recall the relevant definitions [2,4,5]. A contact form on a smooth manifold M of dimension $2n+1 \geq 3$ is a 1-form α such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is everywhere nonzero. The contact structure defined by α is the hyperplane $E(\alpha) \subset TM$ of kernels of α. Two contact forms α_1 and α_2 define the same contact structure if and only if there exists a smooth nowhere zero function ν such that $\alpha_2 = \nu\alpha_1$. For any contact form α, there exists a unique vector field ξ_α, called the characteristic vector field of α, or the Reeb field, such that $i(\xi_\alpha)\alpha = 1$ and $i(\xi_\alpha)\,\mathrm{d}\alpha = 0$, here $i(\cdot)$ stands for the interior product. An *almost contact structure* is a triple (ϕ, ξ, η) where ϕ is a 1–1 tensor field, ξ is a vector field, η a 1-form such that

$$\eta(\xi) = 1, \qquad \phi^2(X) = -X + \eta(X)\xi, \quad \forall X.$$

These conditions imply that $\phi\xi = 0$. The quaternionic analogue of an almost contact structures is an *almost contact* 3*-structure*: that is a set of three almost contact structures (ϕ_i, ξ_i, η_i) such that

$$\eta_i(\xi_j) = \delta_{ij}, \qquad \phi_i\xi_j = \epsilon_{ijk}\xi_k, \qquad \eta_i \circ \phi_j = \epsilon_{ijk}\eta_k,$$
$$\phi_i\phi_j(X) = -\delta_{ij}X + \eta_j(X)\xi_i + \epsilon_{ijk}\phi_k X.$$

Here ϵ_{ijk} is zero when all the symbols are not distinct and if they are it is equal to the signature of the corresponding permutation of the integers 1, 2, 3. See [4].

A triple of contact forms $(\alpha_1, \alpha_2, \alpha_3)$ is called a *contact* 3*-structure* if there exists an almost contact 3-structure (ϕ_i, ξ_i, η_i) such that

$$\alpha_i(\xi_i) > 0, \qquad \mathrm{d}\alpha_i(\phi_i X, \phi_i Y) = \mathrm{d}\alpha_i(X, Y), \quad \forall X, Y.$$

Such an almost contact 3-structure is said to be compatible with $\{\alpha_1, \alpha_2, \alpha_3\}$.

Definition 1. A hypercontact structure on a riemannian manifold (M, g) consists of

$$\{(\alpha_1, \alpha_2, \alpha_3), (\phi_i, \xi_i, \eta_i)_{i=1,2,3}\},$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is a contact 3-structure with a compatible almost contact 3-structure (ϕ_i, ξ_i, η_i) as above and satisfying the following:

$$\eta_i(X) = g(X, \xi_i), \qquad g(X, \phi_i Y) = \mathrm{d}\alpha_i(X, Y),$$
$$g(X, Y) = g(\phi_i X, \phi_i Y) + \eta_i(X)\eta_i(Y).$$

If one can choose $\eta_i = \alpha_i$, then we say that the hypercontact structure is a *contact metric* 3*-structure*. A Sasakian 3-structure is a contact metric 3-structure such that ξ_i are Killing vector fields with respect to the metric g and $[\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k$. We refer to [4] for basic examples of hypercontact structures.

2. The basic instanton

The field $\mathbb{H}$ of quaternions $\{x = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}, x_i \in \mathbb{R}\}$ where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$ and $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$; $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$; $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$ can be identified with $\mathbb{R}^4$ and with $\mathbb{C}^2$. Writing $x = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k} = z_1 + z_2\mathrm{j}$ where $z_1 = x_1 + x_2\mathrm{i}, z_2 = x_3 + x_4\mathrm{i}$ establishes an identification of $\mathbb{H}$ and $\mathbb{C}^2$. The conjugate $\overline{x}$ of a quaternion x is $x_1 - x_2\mathrm{i} - x_3\mathrm{j} - x_4\mathrm{k}$ and $x\overline{x} = \overline{x}x = |x|^2$. Also $\mathbb{H}$ can be viewed as the set of 2×2 complex matrices: $x = z_1 + z_2\mathrm{j}$ corresponds to the matrix

$$\begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix},$$

the determinant of which is the square norm of x. Therefore $SU(2)$ is the group of norm 1 quaternions, i.e. a sphere S^3. Its Lie algebra $su(2)$ is the set of skew hermitian matrices with zero trace. The Pauli matrices:

$$\tau_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

form a basis of $su(2)$. Their commutation relations are:

$$[\tau_1, \tau_2] = 2\tau_3, \qquad [\tau_1, \tau_3] = -2\tau_2, \qquad [\tau_2, \tau_3] = 2\tau_1.$$

Hence $su(2)$ is isomorphic with the imaginary part $\{x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}\}$ of $\mathbb{H}$: we identify τ_1 with i, τ_2 with j and τ_3 with k.

Now $S^7 = \{(p, q) \in \mathbb{H}^2, |p|^2 + |q|^2 = 1\}$, and S^4 is the $\mathbb{H}$ projective line, i.e. the set of equivalence classes $[p, q]$ of elements in $\mathbb{H}^2 - \{0\}$: $(p, q) \sim (p', q')$ iff $p = rp', q = rq'$ for some $r \in \mathbb{H} - \{0\}$.

The tautological bundle assigns to $(p, q) \in S^7$ the equivalence class $[p, q] \in S^4$. This is a principal $SU(2)$ bundle with Pontryagin number $k = +1$. It is easy to see that $\alpha(p, q) = \mathrm{Im}(p\,\mathrm{d}\overline{p} + q\,\mathrm{d}\overline{q})$ is a connection such:

$$\mu^*\alpha = \mathrm{Im}\left(\frac{x\,\mathrm{d}\overline{x}}{1 + |x|^2}\right),$$

where $\mu : \mathbb{R}^4 \to S^7$ is the section over $S^4 - (0, 0, 0, 0, 1) \approx \mathbb{R}^4$:

$$\mu(x) = \frac{(x, 1)}{(1 + |x|^2)^{1/2}}.$$

In other words, α is the basic instanton [1]. See [3, pp. 100–104].

Setting $p = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}, q = y_1 + y_2\mathrm{i} + y_3\mathrm{j} + y_4\mathrm{k}$, and $\alpha = (\alpha_1)\mathrm{i} + (\alpha_2)\mathrm{j} + (\alpha_3)\mathrm{k}$, we have:

$$\alpha_1 = x_2\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_2 + x_4\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_4 + y_2\,\mathrm{d}y_1 - y_1\,\mathrm{d}y_2 + y_4\,\mathrm{d}y_3 - y_3\,\mathrm{d}y_4,$$

$$\alpha_2 = x_3\,dx_1 - x_1\,dx_3 + x_2\,dx_4 - x_4\,dx_2 + y_3\,dy_1 - y_1\,dy_3 + y_2\,dy_4 - y_4\,dy_2,$$

$$\alpha_3 = x_4\,dx_1 - x_1\,dx_4 + x_3\,dx_2 - x_2\,dx_3 + y_4\,dy_1 - y_1\,dy_4 + y_3\,dy_2 - y_2\,dy_3.$$

The radial vector field

$$X = \sum_{n=1}^{4} x_n \partial_{x_n} + y_n \partial_{y_n}$$

satisfies: $i(X)\omega_i = 2\alpha_i$ for $i = 1, 2, 3$, where $\omega_i = d\alpha_i$. By Lemma 3.3 of Geiges–Thomas [4], $\{\alpha_1, \alpha_2, \alpha_3\}$ form a hypercontact structure on S^7, and even a Sasakian 3-structure. Hence we have verified directly the observation of Geiges and Thomas in [4].

Theorem 2. *The basic instanton yields a Sasakian 3-structure on S^7.*

Remark 3. Observe that permutations of coordinates in $\mathbb{R}^8$ exchange the contact forms α_i. Hence these three contact forms define the same contact structure on S^7. It is also worth noting that permutations of coordinates exchange the contact forms above into the contact forms defining the hypercontact structure in Example 2 of [4]. In Theorem 10 we prove that, in general, the contact forms of a Sasakian 3-structure define the same contact structure.

3. General $SU(2)$ connections and contact forms

Consider a principal $SU(2)$ bundle $\pi : P \to M$ over an even-dimensional manifold M, so P is odd-dimensional. Under which conditions the components of a connection along the basis τ_1, τ_2, τ_3 of $su(2)$ are contact forms?

If M is point, then $P = SU(2) \approx S^3$, and we may consider the components of the canonical 1-form θ on $SU(2)$. By definition, $\theta(X) = X$ for all $X \in su(2)$, hence if θ_i are the components along the Pauli matrices τ_i, then $\theta_i(\tau_j) = \delta_{ij}$. Recall the commutation relations of τ_i:

$$[\tau_1, \tau_2] = 2\tau_3, \qquad [\tau_1, \tau_3] = -2\tau_2, \qquad [\tau_2, \tau_3] = 2\tau_1.$$

These commutation relations show that $(\theta_i \wedge d\theta_i)(\tau_1, \tau_2, \tau_3) = 2$ or -2 . Hence each θ_i is a contact form on $SU(2) \approx S^3$.

Soppose now $\dim(M) = 2m \geq 2$, and α is a connection with curvature Ω, then

$$\alpha = \sum_{i=1}^{3} \alpha_i \tau_i, \qquad \Omega = \sum_{i=1}^{3} \Omega_i \tau_i.$$

The equation $\Omega = d\alpha + \frac{1}{2}[\alpha, \alpha]$ reads in components:

$$\Omega_1 = d\alpha_1 + \alpha_2 \wedge \alpha_3, \qquad \Omega_2 = d\alpha_2 + \alpha_3 \wedge \alpha_1, \qquad \Omega_3 = d\alpha_3 + \alpha_1 \wedge \alpha_2.$$

Proposition 4. *If the 2-forms Ω_i are nondegenerate on the horizontal distribution (i.e. the kernel of α), then the 1-forms $\alpha_i, i = 1, 2, 3$, are contact forms.*

Proof. Let ξ_i be the fundamental vector fields defined by τ_i. Then $\alpha(\xi_i) = \tau_i$ hence $\alpha_i(\xi_j) = \delta_{ij}$. Moreover, since ξ_k is a vertical vector field, $i(\xi_k)\Omega = 0$, i.e. $i(\xi_k)\Omega_j = 0\, \forall j$. Hence

$$i(\xi_k)\, \mathrm{d}\alpha_k = i(\xi_k)\Omega_k - i(\xi_k)(\alpha_i \wedge \alpha_j), \quad i, j \neq k.$$

Since $\alpha_i(\xi_j) = \delta_{ij}$, $i(\xi_k)\, \mathrm{d}\alpha_k = 0$. Therefore if α_i were contact forms, the vector fields ξ_i would be their corresponding Reeb fields.

Let $2m$ be the dimension of M and compute:

$$\begin{aligned} \alpha_1 \wedge (\mathrm{d}\alpha_1)^{m+1} &= \alpha_1 \wedge (\Omega_1 - \alpha_2 \wedge \alpha_3)^{m+1} \\ &= (\alpha_1 \wedge \Omega_1 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3) \wedge (\Omega_1 - \alpha_2 \wedge \alpha_3)^m \\ &= (\alpha_1 \wedge \Omega_1^2 - 2\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \wedge (\Omega_1 - \alpha_2 \wedge \alpha_3)^{m-1} \\ &\;\;\vdots \\ &= \alpha_1 \wedge \Omega_1^{m+1} - (m+1)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \Omega^m \\ &= (m+1)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \Omega_1^m. \end{aligned}$$

Hence if B is a basis of the horizontal distribution at a point p, then $\alpha_1 \wedge (\mathrm{d}\alpha_1)^{m+1}(\xi_1, \xi_2, \xi_3, B) = \Omega_1^m(B)$. Hence α_1 is a contact form iff Ω is nondegenerate on the horizontal distribution. A similar calculation works for α_2, α_3 as well. □

Definition 5. A hyperkähler structure on a riemannian manifold (M, g) is a set of three complex structures J_1, J_2, J_3 such that $J_1 J_2 = -J_2 J_1 = J_3$, $J_1 J_3 = -J_3 J_1 = -J_2$, $J_2 J_3 = -J_3 J_2 = J_1$, $g \circ J_i = J_i$ and the 2-forms Ω_i defined by $\Omega_i(X, Y) = g(J_i X, Y)$ are closed (i.e. symplectic forms). Alternatively, we can say that a hyperkähler structure on the riemannian manifold (M, g) is a set of three symplectic forms Ω_i such that there exist three complex structures J_i which leave g invariant, satisfy the quaternion identities and such that $\Omega_i(X, Y) = g(J_i X, Y)$.

Theorem 6. *Let $\pi : P \to M$ be a principal $SU(2)$ bundle and α a connection with curvature Ω and let $\alpha_i, \Omega_i, i = 1, 2, 3$, be the components of α and Ω along the Pauli matrices (basis of $su(2)$). Suppose there is a family on sections $\sigma_j : U_j \to P$ trivializing the bundle (Here $\{U_j\}$ is an open cover over which the bundle is trivial), and smooth nowhere vanishing functions ν_j on U_j such that $\{\nu_j \sigma_j^* \Omega_i\}$, $i = 1, 2, 3$ form a hyperkähler structure on U_j, then $\{\alpha_i\}$ are contact forms on P.*

Corollary 7. *Any (anti)-instanton yields a set of three contact forms.*

Proof. Acoording to [1], see also [3,7], any instanton is gauge equivalent with a connection α whose potential over $U_+ = S^4 - \{p\}$, $p = (0, 0, 0, 0, 1)$ is

$$\mu_+^*\alpha = \operatorname{Im}\left(\frac{(x-a)\,d\bar{x}}{\lambda^2+|x-a|^2}\right),$$

where $a \in \mathbb{H}$ is a quaternionic parameter and λ is a positive real number. Over $U_- = S^4 - \{-p\}$, we have

$$\mu_-^*\alpha = \operatorname{Im}\left(\frac{\lambda^2(x+a)\,d\bar{x}}{1+\lambda^2|x+a|^2}\right),$$

where μ_- is the stereographic projection from the south pole $-p$. Over U_+, U_-, the curvature form Ω reads

$$\Omega_+ = \mu_+^*\Omega = \operatorname{Im}\left(\frac{dx \wedge d\bar{x}}{(\lambda^2+|x-a|^2)^2}\right),$$
$$\Omega_- = \mu_-^*\Omega = \operatorname{Im}\left(\frac{\lambda^2\,dx \wedge d\bar{x}}{(1+\lambda^2|x+a|^2)^2}\right),$$

whose components are $\Omega^i_{+,-} = (K_{+,-})\omega_i$ with:

$$\omega_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4,$$
$$\omega_2 = dx_1 \wedge dx_4 + dx_2 \wedge dx_4,$$
$$\omega_3 = dx_1 \wedge dx_4 - dx_2 \wedge dx_3,$$

and

$$K_+ = -2/(\lambda^2+|x-a|^2)^2, \qquad K_- = -2\lambda^2/(1+|x+a|^2)^2.$$

The three symplectic forms above satisfy: $\omega_i(X,Y) = J_iX \cdot Y$ where $\cdot$ is the usual dot product and J_i are the following complex structures:

$$J_1\partial_1 = -\partial_2, \qquad J_1\partial_3 = \partial_4, \qquad J_2\partial_1 = -\partial_3,$$
$$J_2\partial_2 = \partial_4, \qquad J_3\partial_1 = -\partial_4, \qquad J_3\partial_2 = -\partial_3$$

and obviously these complex structures satisfy the quaternionic identities. Therefore, the hypothesis of our Theorem 6 are satisfied. The corollary follows. □

Remark 8. The symplectic forms $\omega_i, i = 1, 2, 3$, above form a basis of the vector space of self-dual 2-forms on $\mathbb{R}^4$. Hence if α is a self-dual connection and Ω is its curvature, then for any section σ over a trivializing contractible open set U, $(\sigma^*\Omega)_i|U$ is a linear combination of ω_i; in Corollary 7, they are just multiples of ω_i by nowhere vanishing functions. But, in general, $(\sigma^*\Omega)_i|U$ are more complicated. For instance if the Pontryagin number of an $SU(2)$ bundle over S^4 is different from $+1$, or -1, $(\sigma^*\Omega)_i|U$ cannot be a multiple by a nowhere vanishing function of a hypersymplectic structure since according to [4], the Ω_i are not all nondegenerate.

Proof of Theorem 6. In view of Proposition 4, we need only to check that Ω_i are nondegenerate on the horizontal distribution. This is a local problem: we need to check this only over a trivializing open subset U of M. Let $\sigma : U \to P$ be a section and ν a smooth nowhere zero function on U such that $\{\omega_i = \nu\sigma^*\Omega_i\}, i = 1, 2, 3$, form a hypersymplectic structure: i.e. there exists a riemannian metric g on U, three almost complex structures J_i on U satisfying the quaternionic identities (see Definition 5) and such that $g(J_i X, Y) = \omega_i(X, Y)$ and $g \circ J_i = g$.

We now reproduce Geiges–Thomas arguments with small modifications. We denote by $H \subset T(P)$ the horizontal space, i.e. the kernel of α and by $G = \pi^* g$ the pullback of the metric g on $P_U = \pi^{-1}(U)$. If X is a vector field on P, we denote by X_h its horizontal component. If X is horizontal, then $(\sigma_*(\pi_* X))_h = X$ and $\Omega(X_h, \cdot) = \Omega(X, \cdot)$ since Ω vanishes on vertical vectors.

Let now X, Y be two horizontal vector fields on P at $\sigma(x), x \in U$:

$$\begin{aligned}\Omega_i(\sigma(x))(X, Y) &= \Omega_i(\sigma(x))((\sigma_*\pi_* X)_h, (\sigma_*\pi_* Y)_h)\\ &= \Omega_i(\sigma(x))(\sigma_*\pi_* X, \sigma_*\pi_* Y) = (\sigma^*\Omega_i)(x)(\pi_* X, \pi_* Y)\\ &= (1/\nu) g(J_i \pi_* X, \pi_* Y).\end{aligned}$$

This shows that Ω_i are nondegenerate at $H_{\sigma(x)}, x \in U$, since π_* is an isomorphism between the horizontal space at $\sigma(x)$ and the tangent space at $x \in U$.

Any other point $p \in P_U$ has the form $p = \sigma(x) \cdot a = R_a(\sigma(x))$ for some $a \in SU(2)$. If X_p is a horizontal vector field at $p = \sigma(x) \cdot a$, i.e. $X_p \in H_p$, then $X_p = (R_a)_* X_{\sigma(x)}$. Hence for $X_p, Y_p \in H_p$, we have

$$\begin{aligned}\Omega(p)(X_p, Y_p) &= \Omega(R_a(\sigma(x)))((R_a)_* X_{\sigma(X)}, (R_a)_* Y_{\sigma(x)})\\ &= (R_a^*\Omega)(\sigma(x))(X_{\sigma(x)}, Y_{\sigma(x)}).\end{aligned}$$

But the curvature form satisfies $R_a^*\Omega = \mathrm{ad}_{a^{-1}}(\Omega) = a\Omega a^{-1}$. Let (μ_{ij}) be the matrix of $\mathrm{ad}_{a^{-1}} : su(2) \to su(2)$ within the basis τ_1, τ_2, τ_3, then

$$\begin{aligned}\Omega_i(p)(X_p, Y_p) &= \sum_{j=1}^{3} \mu_{ij}\Omega_j(\sigma(x))(X_{\sigma(x)}, Y_{\sigma(x)})\\ &= \sum_{j=1}^{3} \mu_{ij}(1/\nu) g(J_j \pi_* X, \pi_* Y) = (1/\nu) g(\Phi_i \pi_* X, \pi_* Y),\end{aligned}$$

where $\Phi_i = \sum_{j=1}^{3} \mu_{ij} J_j$. Since $\mathrm{ad}_{a^{-1}}$ preserves the natural inner product: $(m, n) = -\frac{1}{2}\mathrm{tr}(m \cdot n)$, the matrix (μ_{ij}) is an orthogonal matrix. This implies that the 1–1 tensors defined on U satisfy the quaternionic identities since the J_i's did. In particular they define complex structures on U depending on $a \in SU(2)$. The equation

$$\Omega_i(p)(X_p, Y_p) = (1/\nu) g(\Phi_i \pi_* X, \pi_* Y)$$

shows that Ω_i are nondegenerate at the horizontal distribution at $\sigma(x) \cdot a$. □

Remark 9.

(1) For $X \in H_p, p = R_a(\sigma(x))$, we define following [4], $\Psi_i X \in H_p$ by $\Psi_i X = ((R_a)_* \Phi_i \pi_* X)_h$. Then $\pi_* \Psi_i X = \Phi_i \pi_* X$. Extending Ψ_i and the riemannian metric G in the vertical direction in the obvious way (like in [4]) into a 1–1 tensor field, and a riemannian metric on P we still denote Ψ_i and G, we get a hypercontact structure $\{G, \xi_i, \alpha_i, \Psi_i\}, i = 1, 2, 3$ on P_U. For instance, any instanton defines a hypercontact structure on $S^7 - S^3 = \pi^{-1}(U_{+,-})$. As Geiges–Thomas observed, this hypercontact structure can be extended to the entire S^7 for the basic (anti)-instanton. The problem for the general (anti)-instanton is that the metrics and the almost complex structures over U_+ and U_- do not match on the intersection when the parameter λ and the quaternionic center a in the formulas for potentials of instantons (see Section 2) are different from 1 (for λ) and 0 for the quaternionic parameter.

(2) In Theorem 2, we have not assumed that the hypersymplectic structures defined on the open sets $\{U_j\}$ are "compatible" so to form an almost quaternion structure on M like in [6]. If this is the case, then the hypercontact structures obtained on $\{\pi^{-1} U_j\}$ fit together into a hypercontact structure on P.

4. Some properties of hypercontact structures

In the notion of a hypercontact structure $\{(\alpha_1, \alpha_2, \alpha_3), (\phi_i, \xi_i, \eta_i)_{i=1,2,3}\}$ on a riemannian manifold (M, g), the ingredients are tied up with strong relations. This suggests that they are not independent. For instance it is well known that in case of hyperkähler manifolds (the even-dimensional version of hypercontact structures), the riemannian metric is determined by the kähler forms, which also determine the three complex structures. Also any linear combination of the three kähler forms is again a kähler form. If they had the same cohomology class, they would be all equivalent by Moser's theorem [8]. We have the analogous result.

Theorem 10. *Let $\{(\alpha_1, \alpha_2, \alpha_3), (\phi_i, \xi_i, \eta_i)_{i=1,2,3}\}$ be a hypercontact structure on a riemannnian manifold (M, g) such that $\alpha_i(\xi_j) = 0, \forall i \neq j$. Then the three contact forms α_i represent the same contact structure.*

Proof. For $t \in [0, 1]$ we want to show that $\omega_t = t\alpha_1 + (1 - t)\alpha_2$ is a contact form. By Gray's stability theorem [5], it follows that there exists a diffeomorphism h and a smooth function ν such that $h^*\alpha_1 = \nu\alpha_2$, i.e. that α_1 and α_2 are equivalent. The same argument shows that α_2 and α_3 are equivalent.

Let H be the g-orthogonal complement to the three-dimensional distribution V spanned by ξ_1, ξ_2, ξ_3. The dimension of M being $4n + 3$, the dimension of H is $4n$. Observe first that

$$d\omega_t(X, Y) = tg(X, \phi_1 Y) + (1 - t)g(X, \phi_2 Y) = g(X, (t\phi_1 + (1 - t)\phi_2)Y).$$

Let ϕ_t denote $(t\phi_1 + (1-t)\phi_2)$, then on H, we have $\phi_t^2 = -(t^2 + (1-t)^2)I_d$, since $\phi_1^2 = \phi_2^2 = -I_d, \phi_1\phi_2 = -\phi_2\phi_1$. Hence $d\omega_t$ is nondegenerate on H. Let $x \in M$ and let $(H_1, \dots, H_{4n})$ be a basis of H_x such that $(d\omega_t)^{2n}(H_1, \dots, H_{4n}) = 1$, we want to show that

$$\omega_t \wedge (d\omega_t)^{2n+1}(x)(\xi_1, \xi_2, \xi_3, H_1, \dots, H_{4n}) \neq 0.$$

The left-hand side of the expression above is a sum (with appropriate signs) of terms of the following type:

$$\omega_t(X_1)(d\omega_t(X_2, X_3)) \cdots (d\omega_t(X_{4n+2}, X_{4n+3})),$$

where $\{X_i\}$ are permutations of $\{\xi_i\}$ and $\{H_j\}$. Any term of the form $d\omega_t(\xi_i, H_j)$ vanishes. Indeed, $d\omega_t(H_j, \xi_i) = tg(H_j, \phi_1\xi_i) + (1-t)g(H_j, \phi_2\xi_i) = 0$ since $\phi_k\xi_i = \epsilon_{kij}\xi_j$ and H is g-orthogonal to the pace V spanned by the ξ_i. Therefore

$$\begin{aligned}\omega_t &\wedge (d\omega_t)^{2n+1}(x)(\xi_1, \xi_2, \xi_3, H_1, \dots, H_{4n}) \\ &= (\omega_t \wedge d\omega_t)(\xi_1, \xi_2, \xi_3)(d\omega_t)^{2n}(H_1, \dots, H_{4n}) \\ &= (\omega_t \wedge d\omega_t)(x)(\xi_1, \xi_2, \xi_3).\end{aligned}$$

We now show that this expression never vanishes. First observe that the vector fields ξ_i are orthonormal:

$$\delta_{ij} = \eta_i(\xi_j) = g(\xi_i, \xi_j).$$

Hence

$$\begin{aligned}d\alpha_1(\xi_1, \xi_2) &= g(\xi_1, \phi_1\xi_2) = g(\xi_1, \xi_3) = 0, \\ d\alpha_1(\xi_1, \xi_3) &= g(\xi_1, \phi_1\xi_3) = -g(\xi_1, \xi_2) = 0, \\ d\alpha_1(\xi_2, \xi_3) &= g(\xi_2, \phi_1\xi_3) = -g(\xi_2, \xi_2) = -1.\end{aligned}$$

Similar arguments show that

$$d\alpha_2(\xi_1, \xi_2) = 0, \qquad d\alpha_2(\xi_1, \xi_3) = 1, \qquad d\alpha_2(\xi_2, \xi_3) = 0.$$

Therefore

$$d\omega_t(\xi_1, \xi_2) = 0, \qquad d\omega_t(\xi_1, \xi_3) = 1 - t, \qquad d\omega_t(\xi_2, \xi_3) = -t.$$

Hence

$$(\omega_t \wedge d\omega_t)(\xi_1, \xi_2, \xi_3) = -t\omega_t(\xi_1) - \omega_t(\xi_2)(1-t).$$

If $\alpha_i(\xi_i) = a_i > 0$, then $\omega_t(\xi_1) = a_1 t, \omega_t(\xi_2) = a_2(1-t)$ and since $\alpha_2(\xi_1) = \alpha_1(\xi_2) = 0$ by hypothesis. We finally have

$$(\omega_t \wedge d\omega_t)(\xi_1, \xi_2, \xi_3) = -(a_1 t^2 + a_2(1-t)^2).$$

The minimum of the parabola (in t variable) $u_t = (a_1 t^2 + a_2(1-t)^2)$ is $m = a_2 - a_2^2/(a_1 + a_2) = a_2^2((1/a_2) - (1/(a_1+a_2))) > 0$ since $a_1 > 0$ and $a_2 > 0$. Hence $(\omega_t \wedge d\omega_t)(\xi_1, \xi_2, \xi_3)$ does not vanish for all $t \in [0, 1]$. This ends the proof of the theorem. □

Remark 11.

(1) The hypothesis in the theorem that $\alpha_i(\xi_j) = 0, \forall i \neq j$, seems a very mild hypothesis which probably can always be satisfied. It is for instance satisfied for contact metric 3-structures (and hence for Sasakian 3-structures).

(2) We suspect that in the definition of hypercontact structure, the riemannian metric is uniquely determined by the contact 3-structure, like the metric in the hyperkähler case is completely determined by the three kähler forms.

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